Comments on "Cosmological viability of $f(R)$-gravity as an ideal fluid and its compatibility with a matter dominated phase (astro-ph/0604431)"

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In the recent paper `astro-ph/0603703` we have shown that $f(R) = R + \mu R^n$ modified gravity dark energy models are not cosmologically viable because during the matter era that precedes the accelerated stage the cosmic expansion is $a \sim t^{1/2}$ rather than $t^{2/3}$. In this short note we wish to comment on the recent paper `astro-ph/0604431` which criticised our results. We show here that the R^n models presented in `astro-ph/0604431` cannot generate a stage with $a \propto t^{2/3}$ *preceding* a stage of accelerated expansion. Hence, though acceptable $f(R)$ dark energy models might exist, the R^n models presented in `astro-ph/0604431` are not viable, confirming our previous results.

Among the various interesting possibilities invoked in order to explain a late-time accelerated expansion, $f(R)$ modified gravity dark energy (DE) models (R is the Ricci scalar) have attracted a lot of attention.

However, we found recently in Ref. [1] that for a large class of $f(R)$ DE models, including R^n models, the usual power-law stage $a(t) \propto t^{2/3}$ *preceding* the late-time accelerated expansion is replaced by a power-law behaviour $a(t) \propto t^{1/2}$. Such an evolution is clearly inconsistent with observations, e.g. the distance to last scattering as measured by CMB acoustic peaks. Hence a viable cosmic expansion history seems to be a powerful constraint on such models.

As the claim of [1] was recently criticised by Capozziello *et al.* (CNOT) [2], we feel it is appropriate to post this short reply and to address explicitly their criticism.

Before that, we would like to make some clarifications. First, it is clear that $f(R)$ gravity models can be perfectly viable in different contexts. Maybe the best example is Starobinsky's model, $f(R) = R + \mu R^2$ [3], which has been the first internally consistent inflationary model. This Lagrangian produces an accelerated stage *preceding* the usual radiation and matter stages. A late-time acceleration in this model requires a positive cosmological constant (or some other form of dark energy) in which case the late-time acceleration is not due to the R^2 term.

Second, it is important to clarify an issue raised in CNOT concerning our work. We checked all our results both in Jordan frame (JF) and Einstein frame (EF), always considering the former as the physical frame (i.e. the frame in which matter is conserved). So the power-law behaviour $a \sim t^{1/2}$ found in JF is neither an artifact nor a pathology of the conformal transformation. It is in fact the same solution of the original Brans-Dicke paper [4] in 1961 (Eq. 60) with $\omega = 0$ (equivalent to $\beta = 1/2$ in our work) and corresponds to solutions found also in other R^n papers such as [5].

CNOT argued that it is possible to have a stage with $a(t) \propto t^{2/3}$ followed by a DE dominated phase for some $f(R)$ models and even for the power-law case $f(R) = \mu R^n$ (notice that we changed the sign of n with respect to our paper in order to match the choice in CNOT) and several examples are suggested. Although we agree that there might be some $f(R)$ models which can be viable (as already stated in [1]), we address here the viability of the specific R^n models suggested in CNOT.

Three types of power-law solutions arising in R^n models are important for our discussion (we assume a flat FRW universe and give all expressions in JF) :

***Solution* A**: the evolution of the scale factor is given by [6]

$$a \propto t^{\alpha_A}\,, \quad \alpha_A = \frac{(1-2n)(1-n)}{2-n}\,. \tag{1}$$

It is an *exact* solution *in the absence of dust*, and an asymptotic solution in the presence of dust. The latter was originally used to give rise to a late-time acceleration for negative n ("curvature dominated late-time attractor") [7, 8]. When $\alpha_A < 0$ the expanding solution is given by $a \propto (t_s - t)^{\alpha_A}$, which corresponds to a phantom solution.

In the phase space (x_1, x_2) with $x_1 \equiv \dot{R}/HR$ and $x_2 \equiv R/H^2$, solution A corresponds to the fixed point

$$(x_1, x_2) = \left(\frac{2(n-2)}{(1-2n)(1-n)}, \frac{6n(4n-5)}{(1-2n)(1-n)} \right)\,. \tag{2}$$

***Solution* B**: the evolution of the scale factor corresponds to

$$a \propto t^{\alpha_B}\,, \quad \alpha_B = 2n/3\,. \tag{3}$$

This solution is obtained in the presence of dust with

$$\Omega_m \equiv \frac{\rho_m}{3H^2F} = -\frac{8n^2-13n+3}{2n^2}\,, \tag{4}$$

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where ρ_m is the energy density of the fluid and $F \equiv \partial f/\partial R = n\mu R^{n-1}$. In the phase space (x_1, x_2), solution B corresponds to

$$(x_1, x_2) = \left(-\frac{3}{n}, \frac{3(4n-3)}{n}\right) . \quad (5)$$

***Solution* C**: this is the so-called ϕ-matter-dominated era (ϕMDE) with scale factor evolution

$$a \propto t^{\alpha_C} , \quad \alpha_C = 1/2 , \quad (6)$$

for any n. This corresponds to the fixed point

$$(x_1, x_2) = (-1/(n-1), 0) . \quad (7)$$

It was shown in [1] that for all n the ϕMDE replaces the usual matter era (after which the universe might fall on the late-time attractor A). CNOT instead pointed out that it is possible to use either solution A or B in order to have a standard matter era ($a \propto t^{2/3}$) followed by an accelerated expansion. Clearly, two possibilities arise: either the universe goes from A to B or vice versa. In the first case A has to behave as a matter era ($\alpha_A = 2/3$), and therefore $n = -0.129$ or $n = 1.295$. In the second case we require the condition $\alpha_B = 2/3$, which corresponds to $n = 1$. Hence the three possible "counter examples" suggested by CNOT are: $n = -0.129$, $n = 1.295$ and $n = 1$. Now we shall investigate whether these cases really provide a viable cosmological evolution.

Let us first analyse the stability of the solutions A and B. Neglecting radiation and considering linear perturbations around the fixed points (x_1, x_2) along the line presented in Ref. [9], we obtain the eigenvalues $\mu_1 = -\frac{5-4n}{1-n}$ and $\mu_2 = -\frac{8n^2-13n+3}{(1-2n)(1-n)}$ for the point A and $\mu_\pm = \frac{3(1-n)\pm\sqrt{(1-n)(-256n^3+608n^2-417n+81)}}{4n(n-1)}$ for the point B. This shows immediately that the case $n = -0.129$ (and values in the vicinity) is excluded because the point A is then stable ($\mu_1 < 0$, $\mu_2 < 0$) and, once reached, it will never give way to acceleration. In other words, the transition from A to B is impossible in this case. When $n = 1.295$ A is a saddle point ($\mu_1 < 0$, $\mu_2 > 0$) and B is a stable spiral. Hence the the transition from A to B is possible, but with this value of n (and values in the vicinity) point B is not accelerated, since then $\alpha_B = 0.863$. It should also be noted that point A corresponds to a solution without matter ($\Omega_m = 0$), so this would be a "matter era" without matter. This leaves as the only possibility $n = 1$ and a transition from B to A.

From Eq. (2) the point A disappears for $n = 1$, which means that the transition from B to A is not possible. As this case merely corresponds to Einstein gravity, it is obvious that one gets the required behaviour $a \propto t^{2/3}$ in the dust-dominated era. However there is no mechanism left for the generation of a late-time acceleration unless some additional DE component is introduced, which is what modified gravity DE models are supposed to avoid.

So we conclude from the discussion above that the solutions suggested in CNOT do not lead to a $a \propto t^{2/3}$ behaviour followed by an accelerated expansion.

Still it may be interesting to consider the scenario with n close to 1, for which $a \propto t^p$ with $p \approx 2/3$, instead of exactly 2/3. Let us study the case with n in the conservative range $0.75 < n < 1.25$, which corresponds to power-law exponents $1/2 < p < 5/6$. Since the point B is a stable spiral for $1 < n < 1.327$, transition from a decelerated matter era to an accelerated one is impossible also in this case. For $0.713 < n < 1$ the point B is a saddle, so a transition is indeed possible. For these values the point A corresponds to a stable node with an (effective) phantom equation of state ($w_{\rm eff} = -1 + \frac{2(2-n)}{3(1-2n)(1-n)} < -1$), whereas the third point C, the ϕMDE, is a stable point as well. Hence the trajectories leaving the point B are attracted either by A or C. We have run our numerical code to investigate the evolution of the system in the space (x_1, x_2). Without including radiation the final attractor is in fact either A or C depending upon initial conditions. However trajectories which are attracted to B first and then finally approach the point A are restricted in a narrow region of phase space. When we start from realistic initial conditions ($|x_1| \ll 1, |x_2| \ll 1$) with inclusion of radiation, the solutions directly approach the fixed point A or C without passing in the vicinity of the point B. In other words we have not found any trajectories in which the radiation era is followed by matter and final accelerated eras. Therefore, our numerical analysis excludes also the range $0.713 < n < 1$, although here, contrary to the cases above, an analytical proof is lacking.

In CNOT the possibility is also mentioned of reconstructing the theory from observations (in particular from the function $H(z)$), in analogy to the reconstruction of scalar-tensor DE models [10]. Clearly we expect that many $f(R)$ DE models can successfully be reconstructed *at low redshifts* for any $H(z)$ corresponding to late-time accelerated expansion. However, nothing guarantees that an $H(z)$ *corresponding to a conventional cosmic expansion at high* z leads to an acceptable $f(R)$. The procedure of reconstruction does not guarantee in fact that the resulting model behaves correctly in the past nor that the solution is stable. Finally, the particular model proposed in CNOT does not contradict our claim on $f(R) = \mu R^n$ models since the numerically reconstructed CNOT $f(R)$ model is not of this type.

In conclusion, CNOT pointed out the possibility to have viable $f(R)$ DE models and we agree that this issue is still open. However the specific R^n examples suggested there confirm, rather than contradict, our claim.

[1] L. Amendola, D. Polarski and S. Tsujikawa, arXiv:astro-ph/0603703.

[2] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi,

arXiv:astro-ph/0604431.
[3] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
[4] C. Brans and H. Dicke, Phys. Rev. **124**, 925 (1961).
[5] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, 123003 (2005).
[6] S. Capozziello, F. Occhionero and L. Amendola, Int. J. Mod. Phys. D **1** (1993) 615.
[7] S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, Int. J. Mod. Phys. D **12**, 1969 (2003).
[8] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004).
[9] E. J. Copeland, M. Sami and S. Tsujikawa, arXiv:hep-th/0603057; B. Gumjudpai, T. Naskar, M. Sami and S. Tsujikawa, JCAP **0506**, 007 (2005).
[10] Boisseau, G. Esposito-Farèse, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000); G. Esposito-Farèse and D. Polarski, Phys. Rev. D **63**, 063504 (2001).